Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH A FINITE NUMBER OF EXCEPTIONAL INFINITESIMAL TRANSFORMATIONS.

By S. D. Zeldin.

1. In a former paper* I have discussed the conditions to be imposed on a finite continuous group with a single exceptional infinitesimal transformation in order that some of the structural constants may be simplified. It is the object of this paper to extend these discussions to groups with any finite number of exceptional infinitesimal transformations.

2. Let the group G be generated by $r+g$ (r, g integers) infinitesimal transformations, and let their corresponding operators be $X_1, X_2, \cdots, X_r, X_{r+1}, \cdots, X_{r+g}$, where

$$X_i = \xi_{i,1}(x_1, \cdots, x_{r+g}) \frac{\partial}{\partial x_1} + \cdots \xi_{i,r+g}(x_1, \cdots, x_{r+g}) \frac{\partial}{\partial x_{r+g}}, \qquad (i = 1, 2, \cdots, r+g).$$

These operators must satisfy the symbolic equation

$$(X_i, X_j) = \sum_{k=1}^{r+g} c_{ijk} X_k \dagger \quad (i, j = 1, 2, \cdots, r+g),$$

where $(X_i, X_j) \equiv X_i X_j - X_j X_i$, and the c_{ijk}'s are the structural constants of the group G. Denoting the operators, corresponding to the transformations of the adjoint of G, by the symbols $E_1, E_2, \cdots, E_r, E_{r+1}, \cdots, E_{r+g}$, where

$$E_i \equiv \sum_{j=1}^{r+g} \sum_{k=1}^{r+g} \alpha_j c_{ijk} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, 2, \cdots, r+g),$$

we must have‡

$$(E_i, E_j) = \sum_{k=1}^{r+g} c_{jik} E_k \qquad (i, j = 1, \cdots, r+g).$$

We shall make the following assumptions about the group G:

(a) It has $g(\leqq r)$ exceptional infinitesimal transformations.

(b) The adjoint of G' (which, we shall show, is isomorphic with G) has one invariant spread.

* "On the Structure of Finite Continuous Groups with a Single Exceptional Infinitesimal Transformation," Dissertation, Clark University, 1917.

† S. Lie, Continuirliche Gruppen, p. 385.

‡ S. Lie, Continuirliche Gruppen, p. 467.

3. For simplicity let us take $X_{r+1}, X_{r+2}, \cdots, X_{r+g}$ to be the g exceptional infinitesimal transformations of G, i.e.,

$$(X_i, X_j) = \sum_{k=1}^{r-g} c_{ijk} X_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix}$$

and

$$(X_i, X_j) = 0 \qquad \begin{pmatrix} i = 1, \cdots, r+g \\ j = r+1, \cdots, r+g \end{pmatrix}.$$

There will then be a linear relationship with constant coefficients between the operators $E_1, \cdots, E_{r+g}$, i.e., we can find constants $\lambda_1, \cdots, \lambda_{r+g}$, not all zero, for which*

$$(A) \qquad \lambda_1 E_1 + \lambda_2 E_2 + \cdots + \lambda_{r+g} E_{r+g} = 0.$$

The solution of equation (A) is evidently $\lambda_1 = \lambda_2 = \lambda_3 = \cdots = \lambda_r = 0$, while $\lambda_{r+1}, \lambda_{r+2}, \cdots, \lambda_{r+g}$ are arbitrary constants. Since

$$E_{r+1} = E_{r+2} = \cdots = E_{r+g} = 0,$$

it follows that

$$(E_i, E_j) = \sum_{k=1}^{r} c_{ijk} E_k \qquad (i, j = 1, 2, \cdots, r).$$

Therefore, if G contains just g exceptional transformations, there exists a group, say G', with r essential parameters generated by r infinitesimal transformations $Y_1, \cdots, Y_r$, where

$$Y_i \equiv \sum_{k=1}^{r} \xi_{ik}(y_1 \cdots y_r) \frac{\partial}{\partial y_k},$$

such that

$$(Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k \qquad (i, j = 1, 2, \cdots, r).$$

We shall denote the operators of the adjoint of G' by the symbols $\mathcal{E}_1, \mathcal{E}_2, \cdots, \mathcal{E}_r$, where

$$\mathcal{E}_i = \sum_{j=1}^{r} \sum_{k=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k}.\dagger$$

4. Since we assumed that the adjoint of G' has just one invariant, it follows that the nullity of the matrix

$$\sum_{i=1}^{r} \alpha_i \mathcal{E}_i \equiv \begin{array}{c} (\Sigma \alpha_i c_{i11}, \ \Sigma \alpha_i c_{i21}, \ \cdots, \ \Sigma \alpha_i c_{ir1}) \\ \left| \begin{array}{ccc} \cdot & \cdot \quad \cdot & \cdot \\ \cdot & \cdot \quad \cdot & \cdot \\ \Sigma \alpha_i c_{i1r}, & \cdots, & \Sigma \alpha_i c_{irr} \end{array} \right| \end{array}$$

* S. Lie, Continuirliche Gruppen, p. 465.

† The α's are the parameters of the group G.

is equal to one,* i.e., at least one of the minors of the determinant $\left|\sum_{i=1}^{r}\alpha_i\mathcal{E}_i\right|$ of order $r-1$ is not zero. But every minor of $\left|\sum_{i=1}^{r}\alpha_i\mathcal{E}_i\right|$ is also a minor of

$$\left|\sum_{i=1}^{r+g}\alpha_i E_i\right| \equiv \begin{vmatrix} \Sigma\alpha_i c_{i11}, & \cdots, & \Sigma\alpha_i c_{i,r,1}, & 0\cdots 0 \\ \vdots & & \vdots & \\ \Sigma\alpha_i c_{i,1,r}, & \cdots, & \Sigma\alpha_i c_{i,r,r}, & 0\cdots 0 \\ \Sigma\alpha_i c_{i,1,r+1}, & \cdots, & \Sigma\alpha_i c_{i,r,r+1}, & 0\cdots 0 \\ \vdots & & \vdots & \\ \Sigma\alpha_i c_{i,1,r+g}, & \cdots, & \Sigma\alpha_i c_{i,r,r+g}, & 0\cdots 0 \end{vmatrix}.$$

Therefore at least one minor of $\left|\sum_{i=1}^{r+g}\alpha_i E_i\right|$ of order $r-1$ is not zero, and thus the nullity of the matrix $\Sigma\alpha_i E_i$ can not exceed $g+1$. Furthermore, for $\alpha_1, \cdots, \alpha_r, \alpha_{r+1}, \cdots, \alpha_{r+g}$ assigned, the symbolic equations

$$(B) \qquad \left(\sum_{i=1}^{r+g}\alpha_i X_i, \sum_{i=1}^{r+g}\eta_i X_i\right) = 0$$

are clearly satisfied for

(1) $\eta_1 = \alpha_1, \cdots, \eta_r = \alpha_r, \eta_{r+1} = \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(2) $\eta_1 = \eta_2 = \cdots = \eta_r = 0, \eta_{r+1} = 1, \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(3) $\eta_1 = \eta_2 = \cdots = \eta_r = \eta_{r+1} = 0, \eta_{r+2} = 1, \eta_{r+3} = \cdots = \eta_{r+g} = 0$

. . . .

($g+1$) $\eta_1 = \cdot \quad \cdots \quad \cdot = \eta_{r+g-1} = 0, \eta_{r+g} = 1$

But from the equations (B) follows the symbolic system of equations

$$\Sigma\alpha_i E_i(\eta_1, \cdots, \eta_r, \cdots, \eta_{r+g})$$

$$\equiv \begin{pmatrix} \Sigma\alpha_i c_{i1}, & \cdots, & \Sigma\alpha_i c_{ir1}, & 0\cdots 0 \\ \cdot & \cdots & \cdot & \cdot \\ \cdot & \cdots & \cdot & \cdot \\ \Sigma\alpha_i c_{i1r+g}, & \cdots, & \Sigma\alpha_i c_{irr+g}, & 0\cdots 0 \end{pmatrix}\!\!(\eta_1, \cdots, \eta_r, \cdots, \eta_{r+g}) = 0.\dagger$$

This system has $g+1$ independent solutions, namely

(1) $\eta_1 = \alpha_1, \eta_2 = \alpha_2, \cdots, \eta_r = \alpha_r, \eta_{r+1} = \cdots = \eta_{r+g} = 0$

(2) $\eta_1 = \cdots = \eta_r = 0, \eta_{r+1} = 1, \eta_{r+2} = \cdots = \eta_{r+g} = 0$

(3) $\eta_1 = \cdots \qquad = \eta_{r+1} = 0, \eta_{r+2} = 1, \eta_{r+3} = \cdots = \eta_{r+g} = 0$

. . . .

. . . .

($g+1$) $\eta_1 = \cdots = \eta_r = \cdot \quad \cdots \quad \cdot = \eta_{r+g-1} = 0, \eta_{r+g} = 1.$

* See proof in my dissertation, Clark University, 1917.

† This notation is taken from A. Cayley; see Philosophic Transactions, v. 148, 1858, pp. 39–46.

Therefore the nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is at least $g + 1$. The nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is thus just equal to $g + 1$.

5. **The number of invariants of the adjoint of G.** Since the nullity of the matrix $\sum_{i=1}^{r+g} \alpha_i E_i$ is equal to $g + 1$, the adjoint of G must have $g + 1$ independent invariants, i.e., the system of r partial differential equations

$$
(C)\qquad
\begin{aligned}
E_1 f(\alpha_1 \cdots \alpha_{r+g}) &\equiv \sum_{j=1}^{r+g} \alpha_j c_{1j1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r+g} \alpha_j c_{1j,\, r+g} \frac{\partial f}{\partial \alpha_{r+g}} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
E_r f(\alpha_1 \cdots \alpha_{r+g}) &\equiv \sum_{j=1}^{r+g} \alpha_j c_{rj1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r+g} \alpha_j v_{1j,\, r+g} \frac{\partial f}{\partial \alpha_{r+g}} = 0,
\end{aligned}
$$

will be satisfied by $g + 1$ independent functions

$$
f = \varphi_1(\alpha_1 \cdots \alpha_{r+g}), \quad \varphi_2(\alpha_1 \cdots \alpha_{r+g}), \quad \cdots, \quad \varphi_{g+1}(\alpha_1 \cdots \alpha_{r+g}).
$$

We have assumed that the adjoint of G' has one invariant, i.e., the system of partial differential equations

$$
(D)\qquad
\begin{aligned}
\mathcal{E}_1 f(\alpha_1 \cdots \alpha_r) &\equiv \sum_{j=1}^{r} \alpha_j c_{1j1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{1jr} \frac{\partial f}{\partial \alpha_r} = 0, \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
\mathcal{E}_r f(\alpha_1 \cdots \alpha_r) &\equiv \sum_{j=1}^{r} \alpha_j c_{rj1} \frac{\partial f}{\partial \alpha_1} + \cdots + \sum_{j=1}^{r} \alpha_j c_{rjr} \frac{\partial f}{\partial \alpha_r} = 0,
\end{aligned}
$$

is satisfied by only one function $f(\alpha_1 \cdots \alpha_r)$, and all the other solutions of (D) can be expressed as linear functions of f. It is clear that $f(\alpha_1 \cdots \alpha_r)$ (the solution of (D)) will also satisfy the system (C), for,

$$
\frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+1}} = \frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+2}} = \cdots = \frac{\partial f(\alpha_1 \cdots \alpha_r)}{\partial \alpha_{r+g}} = 0.
$$

We may state, therefore, the following

THEOREM: *If the adjoint of G' has one invariant, the adjoint of G has then $g + 1$ independent invariants, one of which is the invariant of the adjoint of G'.*

6. **The intersections of the spreads invariant to the adjoint of G.** Suppose that $\varphi_1(\alpha)$ is the function, invariant to both the adjoint of G and the adjoint of G'. Let then $\varphi_2(\alpha) = 0$, $\varphi_3(\alpha) = 0$, $\cdots$, $\varphi_{g+1}(\alpha) = 0$, the remaining g spreads invariant to the adjoint of G, be all flats of order $r + g - 2$ in the $r + g - 1$ space, and let their common intersection

(if there is any) be an $r-1$ flat. This flat will be represented by the equations

$$(F)\qquad \begin{aligned} \varphi_2(\alpha_1\cdots\alpha_{r+g}) &= 0,\\ \varphi_3(\alpha_1\cdots\alpha_{r+g}) &= 0,\\ &\cdots\cdots\\ &\cdots\cdots\\ \varphi_{g+1}(\alpha_1\cdots\alpha_{r+g}) &= 0. \end{aligned}$$

It is evident that this common intersection will also be invariant to the adjoint of G.

Let us consider g points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(i)}, \cdots, \alpha^{(g)}$ in the space of the adjoint of G, the coördinates of $\alpha^{(i)}$ $(i = 1, 2, \cdots, g)$ being all zero except the $(r+i)$th which shall be taken equal to unity. We may take $\alpha^{(i)}$ $(i = 1, 2, \cdots, g)$ to correspond to the invariant subgroups, X_{r+i} $(i = 1, 2, \cdots, g)$ of order one of the group G. Then, in the space of the adjoint of G there will be g points invariant to all the transformations of the adjoint of G.*

If, now, the $(r-1)$ flat (F) does not pass through any of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$, then, by Lie's theorem,† we can take $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r, \overline{X}_{r+1}, \cdots, \overline{X}_{r+g}$ such linear functions of the operators $X_1, X_2, \cdots, X_r, X_{r+1}, \cdots, X_{r+g}$ that

$$\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r$$

will form an invariant subgroup of order r of the adjoint of G, i.e.,

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix}.$$

$$\therefore\ \bar{c}_{ijr+1} = \bar{c}_{ijr+2} = \cdots = \bar{c}_{ijr+g} = 0.$$

7. Let us now suppose that $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$ are not flats but algebraic spreads of orders $m_2, m_3, \cdots, m_{g+1}$ respectively. There will then be g^2 polar flats of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$ taken with respect to $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$. If the spreads $\varphi_2(\alpha), \cdots, \varphi_{g+1}(\alpha)$ do not pass through any of the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$, then the $r-1$ flat, the common intersection (if there is any) of g of the g^2 polar flats, will not pass through any of those points. It will therefore be possible, as we have shown above, to take $\overline{X}_1, \cdots, \overline{X}_{r+g}$ such linear functions of $X_1, \cdots, X_{r+g}$ that $\overline{X}_1, \overline{X}_2, \cdots, \overline{X}_r$ will form an invariant subgroup of G, i.e.,

$$\bar{c}_{ijr+1} = \bar{c}_{ijr+2} = \cdots = \bar{c}_{ijr+g} = 0.$$

* See S. Lie, Continuirliche Gruppen, p. 468.

† S. Lie, Continuirliche Gruppen, p. 479.

8. The case might arise when $\varphi_2(\alpha)$ passes through one of the points $\alpha^{(1)}, \cdots, \alpha^{(g)}$, say through $\alpha^{(1)}$, but not through the others, and $\varphi_3(\alpha)$ passes through $\alpha^{(2)}$, but not through the others, etc. Then the polar flat of $\alpha^{(1)}$ with respect to $\varphi_2(\alpha)$ will also pass through that point; similarly, the polar flat of $\alpha^{(2)}$ quâ $\varphi_3(\alpha)$ will pass through $\alpha^{(2)}$, etc. The common intersection (if there is any) of those g^2 polar flats may or may not pass through all the points $\alpha^{(1)}, \alpha^{(2)}, \cdots, \alpha^{(g)}$. If the common intersection is an $r - 1$ flat not passing through those points, then we come to the case which we have already discussed above. If, however, it does pass through them, then by Lie's theorem, to which reference was made above, we can choose $\overline{X}_1, \cdots, \overline{X}_{r+g}$ such linear functions of $X_1, \cdots, X_{r+g}$ that $\overline{X}_1, \cdots, \overline{X}_r$ form an invariant subgroup of order r of G, g of which are the operators $X_{r+1}, X_{r+2}, \cdots, X_{r+g}$.

We shall then have

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r, r+1, \cdots, r+g \\ j = 1, 2, \cdots, r \end{pmatrix};$$

and assuming, for simplicity, that

$$\overline{X}_{r-g+1} = X_{r+1},\ \overline{X}_{r-g+2} = X_{r+2},\ \cdots,\ \overline{X}_r = X_{r+g},$$

we shall have

$$(\overline{X}_i, \overline{X}_j) = \sum_{r-g+1}^{r} \bar{c}_{ijk}\overline{X}_k \qquad \begin{pmatrix} i = 1, 2, \cdots, r \\ j = r-g+1, r-g+2, \cdots, r \end{pmatrix}.$$

But that means that G' which has the same structural constants as the corresponding $r - 1$ operators of G, has an invariant subgroup of order g, which is contrary to the assumption that the adjoint of G' has one invariant spread, while an invariant subgroup of order g of G' would necessitate an invariant $g - 1$ flat of the adjoint of G.*

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
March 28, 1920.

* S. Lie, Continuirliche Gruppen, p. 479.